ON THE ZEROS OF THE FUNCTION, $P(x)$, COMPLEMENTARY TO THE INCOMPLETE GAMMA FUNCTION*

BY

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Introduction

The problem of locating the real zeros of the function† $P(x)$, defined by the definite integral

$$P(x) \equiv \int_0^1 e^{-t} t^{x-1} \, dt, \qquad x > 0,$$

or by the series

$$P(x) \equiv \sum_{s=1}^{s=\infty} \frac{(-1)^s}{s!} \frac{1}{x+s}$$

has been studied by Bourguet‡ whose results may be stated as follows:

The function $P(x)$ has no real zeros save in the intervals

$$\left.\begin{array}{r} -2n < x < -2n + \frac{1}{2} \\ -2n + \frac{1}{2} < x < -2n + 1 \end{array}\right\}, \; n = 3, 4, 5 \cdots,$$

in each of which it has at least one.

The close relation of $P(x)$ to the gamma function, i. e., $\Gamma(x) = P(x) +$ integral transcendental function, and the fact that $P(x)$ is involved in certain functions used for the representation of statistical frequency distributions,§ make it desirable to complete Bourguet's results.

In this note I prove that the function $P(x)$ has at most two real zeros in each of the intervals

$$-2n < x < -2n + 1, \qquad n = 3, 4, 5, \cdots,$$

and consequently complete Bourguet's theorem so that it reads:

The function $P(x)$ has no real zeros save in the intervals

* Presented to the Society, January 2, 1915.

† Prym, F. E., Journal für Mathematik, vol. 82 (1876), pp. 165–172.

‡ Bourguet, L., Acta Mathematica, vol. 2 (1883), pp. 296–298.

§ Charlier, C. V. L., Meddelande från Lunds Astronomiska Observatorium, no. 26 (1905), p. 6.

$$-2n < x < -2n + \tfrac{1}{2}, \qquad -2n + \tfrac{1}{2} < x < -2n + 1,$$

$$n = 3, 4, 5, \cdots,$$

in each of which it has exactly one.

The proof depends upon an extension of the well-known theorem of Budan-Fourier. This extension though noticed by Stern* and Laguerre† and recently considered by Hurwitz,‡ appears not to have been employed in the literature to an extent commensurate with its merits.

1. **The extended theorem of Budan-Fourier.** In the interval $a \leqq x \leqq b$ let $f(x)$ have at each point a finite derivative of order N and let this derivative be of constant sign in the interval. Then the extended theorem of Budan-Fourier states that

The number of real roots of the equation $f(x) = 0$ which lie in the interval $a < x < b$ cannot be greater than the excess of the number of alternations of sign in the sequence

$$f(a), \quad f'(a), \quad f''(a), \quad \cdots, \quad f^{(N-1)}(a), \quad f^{(N)}(a)$$

over that in the sequence

$$f(b), \quad f'(b), \quad f''(b), \quad \cdots, \quad f^{(N-1)}(b), \quad f^{(N)}(b).$$

If the number of roots is not equal to this excess it falls short thereof by an even number.

The extended theorem may be proved by a method very similar to that used for the more restricted and usual form. The extended theorem is particularly useful in studying functions of the form $f(x) = \phi(x) + \phi_n(x)$, where $\phi_n(x)$ is a polynomial of degree n and $\phi(x)$ is a function whose Nth derivative, $N > n$, is of constant sign in the interval under investigation.

2. **Reduction to the interval** $0 < x < 1$. The function $P(x)$ satisfies the difference-equation§

$$eP(x+1) = xeP(x) - 1.$$

If we consider the interval $-2n < x < -2n + 1$ and put $x = -2n + \theta$

$$P(x) = P(-2n + \theta), \qquad 0 < \theta < 1.$$

Successive applications of the difference equation then give

$$eP(x) = eP(-2n + \theta) = \frac{\{eP(\theta) + W_{2n-1}(\theta)\}}{w_{2n}(\theta)},$$

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* Stern, M. A., Journal für Mathematik, vol. 22.

† Laguerre, E., Acta Mathematica, vol. 4 (1884), p. 114.

‡ Hurwitz, A., Mathematische Annalen, vol. 71 (1911), pp. 584–591.

§ Prym, loc. cit., p. 167.

where

$$w_k(\theta) \equiv (\theta - 1)(\theta - 2) \cdots (\theta - k)$$

and

$$W_k(\theta) \equiv 1 + w_1(\theta) + w_2(\theta) + \cdots + w_k(\theta).$$

Since $w_{2n}(\theta)$ is obviously of constant sign in the interval $0 < \theta < 1$ the zeros of $P(-2n + \theta)$ lying within the interval $0 < \theta < 1$ are identical with those of the function

$$R_{2n}(\theta) \equiv eP(\theta) + W_{2n-1}(\theta)$$

which lie within the same interval.

Now since $\theta > 0$, we may use the integral representation of $P(\theta)$:

$$P(\theta) = \int_0^1 e^{-t}\, t^{\theta-1}\, dt$$

and may differentiate the definite integral under the sign. Hence

$$P'(\theta) = -\int_0^1 e^{-t}\, t^{\theta-1} \left\{ ln\left(\frac{1}{t}\right)\right\}^r dt,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$P^{(r)}(\theta) = (-1)^r \int_0 e^{-t}\, t^{\theta-1} \left\{ ln\left(\frac{1}{t}\right)\right\}^r dt.$$

Consequently $P^{(r)}(\theta)$, $(0 < \theta \leqq 1)$ is of constant sign $(-1)^r$. Since $W_{2n-1}(\theta)$ is a polynomial of degree $2n - 1$,

$$W_{2n-1}^{(2n+s)}(\theta) \equiv 0, \qquad s \geqq 0,$$

and therefore

$$R_{2n}^{(2n+s)}(\theta) = eP^{(2n+s)}(\theta)$$

is of constant sign $(-1)^s$ in $0 < \theta \leqq 1$. We may therefore apply the extended theorem of Budan-Fourier to the function $R_{2n}(\theta)$ and need consider only those derivatives of $R_{2n}(\theta)$ whose orders do not exceed $2n$.

3. **Certain properties of the polynomials** $w_k(\theta)$, $W_k(\theta)$. In order to apply the theorem of Budan-Fourier to the function $R_{2n}(\theta)$ we need information as to the sign and magnitude of the functions $W_k(\theta)$ and their derivatives at $\theta = 1$.

To this end we prove that

$$W_k(1) = 1,$$

$$sgn W_k^{(r)}(1) = (-1)^{k-r}, \qquad 1 \leqq r \leqq k;$$

$$|W_k^{(r)}(1)| > e(r!), \qquad k \geqq 3, \qquad 1 \leqq r \leqq k - 1;$$

$$W_k^{(k)}(1) = k!.$$

Since

$$w_k(\theta) \equiv (\theta-1)(\theta-2)\cdots(\theta-k),$$

we may put

$$w_k(\theta) \equiv a_{k,k}\,\theta^k - a_{k,k-1}\,\theta^{k-1} + \cdots + (-1)^{k-r} a_{k,r}\,\theta^r + \cdots + (-1)^k a_{k,0},$$

where the quantities a_{kr} are evidently positive integers,* and in particular

$$a_{k,k} = 1, \qquad a_{k,k-1} = \frac{k(k+1)}{2}, \qquad a_{k,k-2} = \frac{(k-1)k(k+1)(3k+2)}{24}.$$

As $W_k(\theta)$ is a polynomial of degree k, let us assume

$$W_k(\theta) \equiv A_{k,k}\,\theta^k - A_{k,k-1}\,\theta^{k-1} + \cdots + (-1)^{k-r} A_{k,r}\,\theta^r + \cdots + (-1)^k A_{k,0}.$$

Since

$$W_k(\theta) = 1 + w_1(\theta) + w_2(\theta) + \cdots + w_{k-2}(\theta) + w_{k-1}(\theta) + w_k(\theta),$$

we have

$$W_k(\theta) = W_{k-2}(\theta) + w_{k-1}(\theta) + w_k(\theta),$$

$$W_k(\theta) = W_{k-2}(\theta) + w_{k-2}(\theta)\{\theta - k - 1 + (\theta - k + 1)(\theta - k)\},$$

$$W_k(\theta) = W_{k-2}(\theta) + w_{k-2}(\theta)\{\theta^2 - 2(k-1)\theta + (k-1)^2\};$$

whence

$$A_{k,k} = a_{k-2,k-2} = 1,$$

$$A_{k,k-1} = 2(k-1)\,a_{k-2,k-2} + a_{k-2,k-3} = \frac{(k-1)(k+2)}{2},$$

$$A_{k,k-2} = A_{k-2,k-2} + (k-1)^2 a_{k-2,k-2} + 2(k-1)\,a_{k-2,k-3} + a_{k-2,k-4}$$

$$= 1 + (k-1)^3 + \frac{(k-3)(k-2)(k-1)(3k-4)}{24},$$

. . . .

$$A_{k,r} = A_{k-2,r} + (k-1)^2 a_{k-2,r} + 2(k-1)\,a_{k-2,r-1} + a_{k-2,r-2},$$

. . . .

$$A_{k,2} = A_{k-2,2} + (k-1)^2 a_{k-2,2} + 2(k-1)\,a_{k-2,1} + a_{k-2,0},$$

$$A_{k,1} = A_{k-2,1} + (k-1)^2 a_{k-2,1} + 2(k-1)\,a_{k-2,0},$$

$$A_{k,0} = A_{k-2,0} + (k-1)^2 a_{k-2,0}.$$

From this we see at once that the quantities $A_{k,r}$ are positive integers. $A_{k,k} = 1$, and evidently if $k \geqq 3$, $A_{k,k-1}$ and $A_{k,k-2}$ are greater than 3. Also

$$A_{k,r} > A_{k-2,r}, \qquad 0 \leqq r \leqq k-3.$$

* They are, in fact, the well-known factorial coefficients or Stirling's numbers of the first kind.

Now

$$W_3(\theta) = \theta^3 - 5\theta^2 + 9\theta - 4,$$

$$W_4(\theta) = \theta^4 - 9\theta^3 + 30\theta^2 - 41\theta + 20.$$

Hence we have

$$A_{k,k} = 1, \qquad A_{k,r} > 3 > e, \qquad k \geqq 3, \qquad r \leqq k - 1.$$

From the definition of $w_k(\theta)$ and $W_k(\theta)$ we obtain

$$w_k(\theta + 1) = \theta w_{k-1}(\theta),$$

$$W_k(\theta + 1) = 1 + \theta W_{k-1}(\theta),$$

from which we have

$$\begin{aligned}
W_k'(\theta + 1) &= W_{k-1}(\theta) + \theta W_{k-1}'(\theta),\\
W_k''(\theta + 1) &= 2W_{k-1}'(\theta) + \theta W_{k-1}''(\theta),\\
&\cdots\cdots\cdots\\
W_k^{(r)}(\theta + 1) &= rW_{k-1}^{(r-1)}(\theta) + \theta W_{k-1}^{(r)}(\theta),\\
&\cdots\cdots\cdots\\
W_k^{k-1}(\theta + 1) &= (k-1)W_{k-1}^{(k-2)}(\theta) + \theta W_{k-1}^{(k-1)}(\theta),\\
W_k^k(\theta + 1) &= kW_{k-1}^{(k-1)}(\theta);
\end{aligned}$$

and therefore, since

$$\begin{aligned}
W_k^{(r)}(0) &= (-1)^{k-r} A_{k,\,r} \cdot (r!),\\
W_k(1) &= 1,\\
W_k'(1) &= W_{k-1}(0) = (-1)^{k-1} A_{k-1,\,0},\\
W_k''(1) &= 2W_{k-1}'(0) = (-1)^{k-2} A_{k-1,\,1} \cdot (2!),\\
&\cdots\cdots\cdots\\
W_k^{(r)}(1) &= rW_{k-1}^{(r-1)}(0) = (-1)^{k-r} A_{k-1,\,r-1} \cdot (r!),\\
&\cdots\cdots\cdots\\
W_r^{(k-1)}(1) &= (k-1)W_{k-1}^{(k-2)}(0) = -A_{k-1,\,k-2} \cdot \{(k-1)!\},\\
W_k^{(k)}(1) &= kW_{k-1}^{(k-1)}(0) = k!.
\end{aligned}$$

From this we have at once the desired relations

$$W_k(1) = 1, \qquad \operatorname{sgn} W_k^{(r)}(1) = (-1)^{k-r}, \qquad 1 \leqq r \leqq k; \qquad W_k^{(k)}(1) = k!.$$

Since $A_{k-1,r} > 3 > e$, $k \geqq 4$, $0 \leqq r \leqq k - 2$, we have also

$$|W_k^{(r)}(1)| > e \cdot (r!),. \qquad 1 \leqq r \leqq k - 1.$$

4. **Certain properties of $P(\theta)$.** Since

$$P(\theta) = \int_0^1 e^{-t} t^{\theta-1} dt,$$

and

$$P^{(r)}(\theta) = (-1)^r \int_0^1 e^{-t} t^{\theta-1} \left\{ ln\left(\frac{1}{t}\right) \right\}^r dt,$$

it follows that

$$P(1) = 1 - e^{-1},$$

$$P'(1) = -\int_0^1 e^{-t} ln\left(\frac{1}{t}\right) dt = li(e^{-1}) - C*$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$P^{(r)}(1) = (-1)^r \int_0^1 e^{-t} \left\{ ln\left(\frac{1}{t}\right) \right\}^r dt.$$

Now

$$\frac{r!}{e} = \frac{1}{e}\int_0^1 \left\{ ln\left(\frac{1}{t}\right) \right\}^r dt < \int_0^1 e^{-t} \left\{ ln\left(\frac{1}{t}\right) \right\}^r dt < \int_0^1 \left\{ ln\left(\frac{1}{t}\right) \right\}^r dt = r!.$$

Hence

$$eP(1) = e - 1 = 1.71828 \cdots,$$

$$eP'(1) = -e\{C - li(e^{-1})\} = -2.16538 \cdots,$$

$$\operatorname{sgn} P^{(r)}(\theta) = (-1)^r, \qquad 0 < \theta \leqq 1;$$

$$r! < |eP^{(r)}_{(1)}| < e(r!), \qquad 1 < r.$$

As θ approaches zero $P^{(r)}(\theta)$ becomes infinite of sign $(-1)^r$.

5. **Application of the extended theorem of Budan-Fourier.** Consider now the function

$$R_{2n}(\theta) \equiv eP(\theta) + W_{2n-1}(\theta)$$

and its successive derivatives

$$R'_{2n}(\theta), \qquad R''_{2n}(\theta), \qquad \cdots, \qquad R^{(2n)}_{2n}(\theta).$$

When n has been fixed we can assign a positive quantity A which shall exceed the maximum of the absolute value of the polynomial $W_{2n-1}(\theta)$ and that of the absolute value of each of its derivatives in the interval $0 \leqq \theta \leqq 1$. We may then take θ_0 so small that $|P^{(r)}(\theta_0)| > A$, $0 \leqq r \leqq 2n$. Evidently $R_{2n}(\theta)$ has no roots in the interval $0 \leqq \theta \leqq \theta_0$.

The signs of the terms of the sequence

$$R_{2n}(\theta_0), \qquad R'_{2n}(\theta_0), \qquad \cdots, \qquad R^{(2n)}_{2n}(\theta_0)$$

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* Here $li(e^{-1})$ is the integral-logarithm, and C is Euler's constant, $0.57722 \cdots$. Cf. Nielsen, *Theorie des Integrallogarithmus*, pp. 2, 11, 89.

are then evidently the same as those of the sequence (of $2n + 1$ terms)

$$P(\theta_0), \quad P'(\theta_0), \quad \cdots \quad P^{(r)}(\theta_0) \quad \cdots \quad P^{(2n-1)}(\theta_0), \quad P^{(2n)}(\theta_0),$$

and these are alternately positive and negative. There are therefore $2n$ alternations of sign in this sequence.

On passing to the other end of the interval we have

$$R_{2n}(1) = eP(1) + W_{2n-1}(1) = +e,$$

$$R'_{2n}(1) = eP'(1) + W'_{2n-1}(1) = +\{A_{2n-2,0} - 2.16538\cdots\},$$

$$R''_{2n}(1) = eP''(1) + W''_{2n-1}(1) = -\left\{A_{2n-2,1} - \frac{e|P''(1)|}{2!}\right\}(2!),$$

. . . .

$$R^{(r)}_{2n}(1) = eP^{(r)}(1) + W^{(r)}_{2n-1}(1) = (-1)^{r-1}\left\{A_{2n-2,r-1} - \frac{e|P^{(r)}(1)|}{r!}\right\}(r!),$$

. . . .

$$R^{(2n-2)}_{2n}(1) = eP^{(2n-2)}(1) + W^{(2n-2)}_{2n-1}(1)$$

$$= -\left\{A_{2n-2,2n-3} - \frac{e|P^{(2n-2)}(1)|}{(2n-2)!}\right\}(2n-2)!,$$

$$R^{(2n-1)}_{2n}(1) = eP^{(2n-1)}(1) + W^{(2n-1)}_{2n-1}(1) = -\left\{\frac{e|P^{(2n-1)}(1)|}{(2n-1)!} - 1\right\}(2n-1)!,$$

$$R^{(2n)}_{2n}(1) = eP^{(2n)}(1) = +|eP^{2n}_{(1)}|.$$

Now if $n \geqq 3$, $2n - 2 \geqq 4$. Hence

$$A_{2n-2,0} > 3 > 2.16538\cdots, \quad A_{2n-2,\,r-1} > e > \frac{e|P^{(r)}(1)|}{r!}, \quad (1 \leqq r \leqq 2n-2),$$

but

$$\frac{e|P^{(2n-1)}(1)|}{(2n-1)!} - 1 > 0.$$

Hence the sequence of signs in

$$R_{2n}(1), \quad R'_{2n}(1), \quad R''_{2n}(1), \quad \cdots \quad R^{(r)}_{2n}(1), \quad \cdots \quad R^{(2n-3)}_{2n}(1), \quad R^{(2n-2)}_{2n}(1),$$

$$R^{(2n-1)}_{2n}(1), \quad R^{(2n)}_{2n}(1)$$

is

$$+ \quad + \quad - \quad \cdots \quad (-1)^{r-1} \cdots \quad + \quad - \quad - \quad +$$

As compared with the sequence of signs at $\theta = \theta_0$ we see that this sequence has lost two alternations and only two, namely, one between R and R', and one between $R^{(2n-2)}$ and $R^{(2n-1)}$.

Hence $R_{2n}(\theta)$ can have at most two zeros in the interval $\theta_0 \leqq \theta \leqq 1$ and therefore at most two in the interval $0 < \theta < 1$.

Therefore $P(x)$ can have at most two zeros in the interior of the interval

$$-2n < x < -2n + 1;$$

which is the theorem to be proved.

DARTMOUTH COLLEGE,
December 14, 1914.